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Probabilistic limit analysis and lifetime prediction of concrete structures

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This paper presents a general approach to the probabilistic prediction of the lifetime of reinforced concrete frames with respect to structural collapse. The structural system is considered to be exposed to an aggressive environment and the effects of the structural damaging process are described by the corresponding evolution in time of the axial force–bending moment resistance domains. The collapse load is computed by means of limit analysis. Monte Carlo simulations are used to account for the randomness of the main structural parameters. In this way, both the time-variant probability of failure, as well as the expected structural lifetime associated with a prescribed reliability level, are evaluated. An application to the probabilistic time-variant limit analysis and lifetime prediction of a reinforced concrete arch bridge is presented.

Keywords: Concrete structures; Aggressive environments; Structural damage; Lifetime prediction; Limit analysis

1. Introduction

The safety level associated with the ultimate limit state of structural collapse can be adequately evaluated only after suitable structural models, which are able to describe the fundamental behaviour of the structural system, have been selected. In many cases, this aim can be achieved by assuming perfectly plastic behaviour and neglecting second order effects, which make the general theory of limit analysis applicable to steel structures. In spite of such idealizations, this theory can also be successfully applied to concrete structures, at least for the prediction of the collapse loads, if the concrete tensile strength is neglected and the concrete compression strength is properly modified through a suitable effectiveness factor (Nielsen 1999).

However, for concrete structures the structural performance must be considered as time-dependent, mainly because of the progressive deterioration of the mechanical properties of materials that makes the structural system less able to withstand the applied actions. Therefore, in order to ensure an adequate level of structural performance during the whole service life of the structure, the structural model must be also able to account for structural deterioration.

Starting from the previous considerations, a systematic approach to the limit analysis of plane framed structures, which considers axial force and bending moment as active and interacting generalized plastic stresses is considered (Biondini 2000). The structural system is considered to be exposed to an aggressive environment and the effects of the damaging process are described by the corresponding evolution of the axial force–bending moment resistance domains. In particular, such evolution is obtained by means of a proper methodology recently proposed for the durability analysis and lifetime assessment of concrete structures subjected to a diffusive attack from external aggressive agents (Biondini *et al.* 2004). The complete solution of the problem (i.e. the collapse loads, a stress distribution at the incipient collapse and a collapse mechanism) is then obtained at each time instant by linear programming.

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The solution provided by the proposed time-variant limit analysis procedure makes the importance of environmental damage very clear from a qualitative point of view. However, due to the uncertainties in material and geometrical properties, in the magnitude and distribution of the loads, in the physical parameters that define the deterioration process, among others, the previous deterministic results cannot be used for reliable quantitative predictions. For this reason, the collapse loads must be considered as random variables or processes and the time-variant structural safety can be realistically assured only in probabilistic terms. This problem can be solved by using Monte Carlo simulations (Biondini 2000, Biondini *et al.* 2006). In this way, the probability density distribution of the collapse load multiplier is derived at each time instant and the corresponding time-variant probability of failure, as well as the expected structural lifetime associated with a prescribed reliability level, are evaluated. A final application to the probabilistic time-variant limit analysis and lifetime prediction of a reinforced concrete arch bridge is presented.

2. Limit analysis of framed structures

A systematic approach to the limit analysis of plane framed structures is presented here. The proposed approach neglects shear failures, which can be avoided by a proper capacity design, and considers the cross-sectional axial force n and bending moment m as active and interacting generalized plastic stresses. Following the general theory of limit analysis, a *rigid perfectly-plastic constitutive law* is adopted to relate these stresses to the correlative generalized plastic strains, represented by the cross-sectional axial elongation Δl and bending rotation θ, respectively. In this way, the behaviour of the discrete cross-sections where the plastic strains tend to develop can be represented by a *generalized plastic hinge* that allows a free axial-bending kinematic behaviour and, contemporaneously, fully transfers the corresponding plastic values of the axial force and bending moment. The plastic collapse under proportionally increasing loads is reached when the set of generalized plastic hinges is able to activate a kinematic mechanism for which the equilibrium can no longer be satisfied.

2.1 *Equilibrium and compatibility conditions*

Forces and generalized stresses are assumed in accordance with the conventions and with the reference systems shown in figure 1 (Biondini 2000). In the following, *equilibrium* and *compatibility* conditions are derived on the basis of the classical *small displacements hypothesis*.

The end forces and the internal generalized stress of a beam element can be posed as a function of the applied loads and of three independent statical quantities (Livesley 1975). Since it is reasonable to replace a distributed load with statically equivalent concentrated loads in an appropriate number of cross-sections, the element is considered subjected only to concentrated forces normal to the beam axis. When axial loads or applied moments are present, the beam element can be further subdivided. In this way, it is convenient to select as reference quantities the axial force n and the end moments $\bar{m}_1 = -m_{z'1}$ and $\bar{m}_2 = +m_{z'2}$ (figure 2). Thus, in the local coordinate system (x', y') the following relationships hold:

$$\mathbf{f}'_i = \mathbf{H}'_i \mathbf{r} - \mathbf{f}^{e'}_i, \quad i = 1, 2, \tag{1}$$

with $\mathbf{f}'_i = [n_{x'i} \quad t_{y'i} \quad m_{z'i}]^T$, $\mathbf{r} = [n \ \bar{m}_1 \ \bar{m}_2]^T$, and where the equivalent nodal force vectors $\mathbf{f}^{e'}_i$ and the equilibrium matrices $\mathbf{H}'_i$ are obtained from the equilibrium conditions (figure 2):

$$\mathbf{H}'_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/l & 1/l \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{H}'_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/l & -1/l \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

$$\begin{aligned} \mathbf{f}^{e'}_1 &= [0 \quad 1 \quad 0]^T \sum_j f_{y'j}(1 - l_j/l), \\ \mathbf{f}^{e'}_2 &= [0 \quad 1 \quad 0]^T \sum_j f_{y'j}(l_j/l). \end{aligned} \tag{3}$$

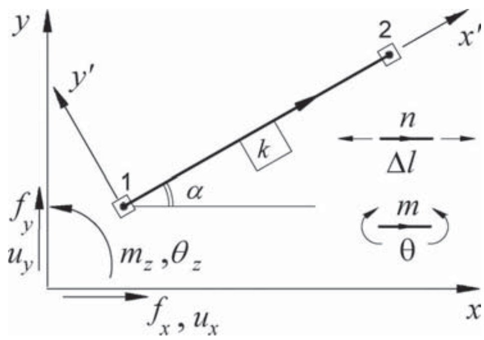

Figure 1. Reference systems.

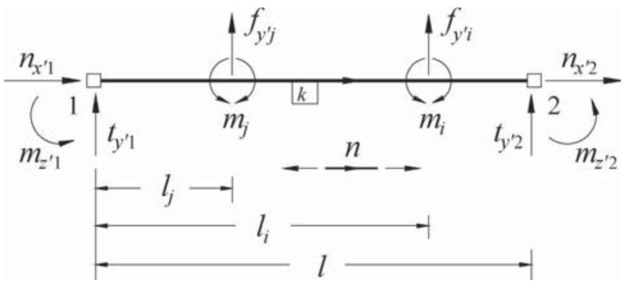

Figure 2. Reference statical quantities.

The previous equations can be written in the global reference system (x, y) by means of a coordinate transformation matrix $\mathbf{T}=\mathbf{T}(\alpha)$:

$$\mathbf{f}_i = \mathbf{T}\mathbf{f}'_i = \mathbf{T}(\mathbf{H}'_i\mathbf{r} - \mathbf{f}_i^{e'}) = \mathbf{H}_i\mathbf{r} - \mathbf{f}_i^e, \quad i = 1, 2, \tag{4}$$

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

In this way, by denoting with $\mathbf{f}_h^0$ the load vector applied at the node h of the structure in the global reference system, the nodal equilibrium conditions can be written as:

$$\mathbf{f}_h^0 = \sum_{k\to h} \mathbf{f}_{ik}, \tag{6}$$

or,

$$\mathbf{f}_h^0 + \sum_{k\to h} \mathbf{f}_{ik}^e = \sum_{k\to h} \mathbf{H}_{ik}\mathbf{r}_k, \tag{7}$$

where the sums are over all the elements k whose end i converges at the node h. Finally, by assembling over all the nodes h, the nodal equilibrium equations for the whole structure can be synthesized as:

$$\mathbf{f}_A = \mathbf{H}_A\mathbf{r}_A. \tag{8}$$

Apart from the ends, the more critical cross-sections of the element are those directly loaded. The bending moments in each cross-section i can also be expressed as a function of the applied loads:

$$m_i^0 = \mathbf{h}_i^T\mathbf{r} + m_i, \tag{9}$$

where $\mathbf{h}_i$ and m_i^0 are obtained again by simple equilibrium (figure 2):

$$\mathbf{h}_i^T = -[0 \quad 1 - l_i/l \quad l_i/l] \tag{10}$$

$$m_i^0 = \sum_{j<i} f_{y'j}(l_i - l_j) - (l_i/l)\sum_j f_{y'j}(l - l_j). \tag{11}$$

The previous equilibrium equation does not depend on the reference system. After assembling over all the loaded cross-sections i, the internal equilibrium equations for the whole structure can be synthesized as:

$$\mathbf{f}_B = \mathbf{H}_B\mathbf{r}_A + \mathbf{r}_B. \tag{12}$$

In conclusion, the generalized stress vector $\mathbf{r} = [\mathbf{r}_A^T \quad \mathbf{r}_B^T]^T$ can be directly related to the load vector $\mathbf{f} = [\mathbf{f}_A^T \quad \mathbf{f}_B^T]^T$, through the following equilibrium matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_A & \mathbf{0} \\ \mathbf{H}_B & \mathbf{I} \end{bmatrix}, \tag{13}$$

or,

$$\mathbf{f} = \mathbf{H}\mathbf{r}. \tag{14}$$

The generalized strains corresponding to the stresses n, $\bar{m}_1$, $\bar{m}_2$, m_i, are the elongation Δl, and the rotations $\bar{\theta}_1$, $\bar{\theta}_2$, θ_i, respectively. Therefore, the stress vector $\mathbf{r}$ can be associated with the strain vector $\mathbf{e} = [\mathbf{e}_A^T \quad \mathbf{e}_B^T]^T$, with $\mathbf{e}_A = [\mathbf{e}_1^T \quad \mathbf{e}_2^T \quad \ldots]^T$, $\mathbf{e}_k = [\Delta l \quad \bar{\theta}_1 \quad \bar{\theta}_2]^T$, $\mathbf{e}_B = [\theta_1 \; \theta_2 \ldots]^T$. In an analogous way, the load vector $\mathbf{f}$ is related to a vector of displacements $\mathbf{s} = [\mathbf{s}_A^T \quad \mathbf{s}_B^T]^T$, with $\mathbf{s}_A = [\mathbf{s}_1^T \quad \mathbf{s}_2^T \quad \ldots]^T$, $\mathbf{s}_k = [u_x \; u_y \; \theta_z]^T$, $\mathbf{s}_B = \mathbf{e}_B$. It can be verified that, between displacements $\mathbf{s}$ and strains $\mathbf{e}$, the following relationship holds (Livesley 1975):

$$\mathbf{e} = \mathbf{H}^T\mathbf{s}. \tag{15}$$

2.2 *Yield conditions and flow rule*

According to the hypothesis of the *rigid perfectly-plastic constitutive law*, (a) the *yielding criterion*, which defines the stress state corresponding to the start of the plastic flow, is convex, and (b) the *flow rule*, through which the increments of the plastic strains are correlated to the stress state, is associated with the yielding surface (normality rule).

By assuming the normal force n and the bending moment m as the only active generalized plastic stresses (figure 3(a)), the yielding criterion for the generic critical cross-section i can be written as $f_i(n_i, m_i) = 0$. Such a criterion defines, in the $n-m$ plane, a curve that can be reasonably idealized by a stepwise approximation, which is, for the sake of safety, inscribed within the convex domain $f_i(n_i, m_i) \le 0$ (figure 3(b)). Therefore, by assuming a stepwise linearization with q_i sides, the yielding criterion for each critical cross-section i is rewritten as:

$$\boldsymbol{\phi}_i = \mathbf{N}_i\mathbf{r}_i - \mathbf{k}_i \le \mathbf{0}, \tag{16}$$

where $\boldsymbol{\phi}_i = [\phi_1^i \quad \phi_2^i \quad \ldots \quad \phi_{q_i}^i]^T$, $\mathbf{N}_i = [\mathbf{n}_1^i \quad \mathbf{n}_2^i \quad \ldots \quad \mathbf{n}_{q_i}^i]^T$, $\mathbf{n}_j^i = [n_{j1}^i \quad n_{j2}^i]^T$, $\mathbf{k}_i = [k_1^i \quad k_2^i \quad \ldots \quad k_{qi}^i]^T \ge \mathbf{0}$, and $\mathbf{r}_i = [n_i \; m_i]^T$ (figure 3(c)). Finally, by assembling these conditions for the whole structure:

$$\boldsymbol{\phi} = \mathbf{N}\mathbf{r} - \mathbf{k} \le \mathbf{0}. \tag{17}$$

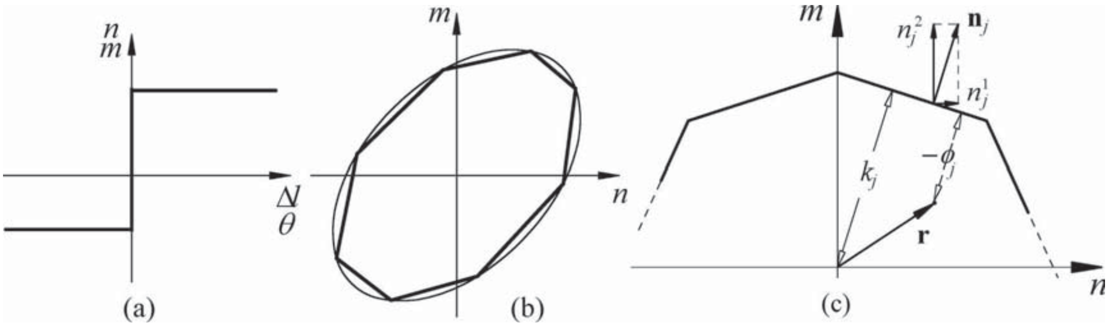

Figure 3. (a) Rigid plastic constitutive laws, (b) yielding curve with its stepwise linearization, and (c) flow rule.

The associated flow rule for each critical cross-section i is given by:

$$\Delta l_i = \mu_i \frac{\partial f_i}{\partial n_i}, \quad \theta_i = \mu_i \frac{\partial f_i}{\partial m_i}, \tag{18}$$

with the multiplier $\mu_i \geq 0$ that allows plastic flows only for the points lying on the yielding curve along the outside normal (figure 3(c)), that is $\mu_i f_i(n_i, m_i) = 0$. For the linearized case:

$$\mathbf{e}_i = \mathbf{N}_i^T \boldsymbol{\mu}_i, \tag{19}$$

with $\phi_j^i \mu_j^i = 0$ $(j = 1,\ldots, q_i)$ and where $\mathbf{e}_i = [\Delta l_i \;\; \theta_i]^T$, $\boldsymbol{\mu}_i = [\mu_1^i \;\; \mu_2^i \;\; \ldots \;\; \mu_{q_i}^i]^T \geq \mathbf{0}$. Summing up the axial strains Δl_i of all the critical cross-sections i of the element k:

$$\Delta l = \sum_{i \in k} \Delta l_i, \tag{20}$$

the plastic flow equations for the whole structure can be assembled as follows:

$$\mathbf{e} = \mathbf{N}^T \boldsymbol{\mu}. \tag{21}$$

2.3 *Static and kinematic approach (duality)*

Let $\mathbf{f}_0$ be a vector of constant loads and $\mathbf{f}$ a vector of loads whose intensity is proportional to a given scalar multiplier $\lambda \geq 0$. By assuming that the structure is safe for $\lambda = 0$, the *collapse multiplier* λ_c associated with its failure is derived from the two fundamental theorems of limit analysis. The *lower bound theorem* states that λ_c is the maximum of the multipliers associated with stress fields that satisfy both the equilibrium conditions and the yielding criterion. In a similar manner, the *upper bound theorem* states that λ_c is the minimum of the multipliers associated with plastic flows that satisfy both compatibility conditions and flow rule. Therefore, in mathematical terms, the fundamental theorems of the limit analysis can be translated into the following dual linear programming problems:

$$\max\{\lambda | (\lambda \mathbf{f} - \mathbf{H}\mathbf{r}) = -\mathbf{f}_0, \mathbf{N}\mathbf{r} \leq \mathbf{k}, \lambda \geq 0\}, \tag{22}$$

$$\min\left\{\mathbf{k}^T \boldsymbol{\mu} - \mathbf{f}_0^T \mathbf{s} \,|\, (\mathbf{N}^T \boldsymbol{\mu} - \mathbf{H}^T \mathbf{s}) = \mathbf{0},\, \mathbf{f}^T \mathbf{s} = 1,\; \boldsymbol{\mu} \geq \mathbf{0}\right\}. \tag{23}$$

It is worth noting that in the second case, the minimum condition is related to the work done by the proportional loads $\mathbf{f}$ for the displacements $\mathbf{s}$ associated with the collapse mechanism. Since this mechanism is associated with an arbitrary multiplier, it results in being univocally identified by the condition $\mathbf{f}^T \mathbf{s} = 1$.

The solution of the previous dual linear programs leads to the *complete solution* of the problem, i.e. the collapse multiplier, the stress distribution at the incipient collapse and the collapse mechanism. As it is the separator of two sets, λ_c is unique. However, the uniqueness of λ_c does not necessarily mean the uniqueness of the collapse mechanism, or that of the stress field at collapse.

2.4 *Time-variant limit analysis*

The previous limit analysis refers to prescribed states of the structure, where the cross-sectional performance is quantified by using fixed values of the quantities $\mathbf{N}$ and $\mathbf{k}$, which define the yielding criterion and the flow rule. However, due to the progressive deterioration of the mechanical properties of materials, such quantities, as well as the corresponding collapse multiplier λ_c, vary during time. In order to account for such variability, a structural analysis of the deteriorating cross-sections is firstly required to build the functions $\mathbf{N} = \mathbf{N}(t)$ and $\mathbf{k} = \mathbf{k}(t)$. In this way, a time-variant limit analysis leading to the time evolution of the collapse multiplier $\lambda_c = \lambda_c(t)$ can also be performed by solving the previous linear programs at several time instants.

3. Lifetime performance of deteriorating concrete cross-sections

Here, attention is focussed on the damaging process induced by the diffusive attack of environmental aggressive agents, like sulphate and chloride. In this context, the diffusion process may lead to the deterioration of the concrete and the corrosion of the reinforcement. In addition, damage induced by mechanical loading interacts with the environmental factors and accelerates the deterioration process (CEB 1992). For these reasons, a reliable tool for the assessment of the time-variant performance of concrete structures in aggressive environments should be capable of accounting for both the diffusion process and the corresponding mechanical damage, as well as for the coupling effects between diffusion, damage, and structural behaviour.

3.1 *Simulation of the diffusion process*

The simplest model to describe the kinetic process of the diffusion of chemical components in solids is represented by Fick's laws, which, in the case of a single component diffusion in isotropic, homogeneous and time-invariant media, can be reduced to the following second order partial differential linear equation:

$$D\nabla^2 C = \frac{\partial C}{\partial t}, \qquad (24)$$

where D is the diffusivity coefficient of the medium, $C = C(\mathbf{x}, t)$ is the concentration of the aggressive agent at point $\mathbf{x}$ and time t, $\nabla C = \text{grad}\ C(\mathbf{x}, t)$ and $\nabla^2 = \nabla \cdot \nabla$.

From the numerical point of view, the diffusion equation can be effectively simulated by using cellular automata that, in their basic form, consists of regular uniform grids of *sites* or *cells*, theoretically having an infinite extension, with a discrete variable in each cell that can take on a finite number of states (Wolfram 1994). In particular, Fick's laws in two dimensions can be accurately reproduced by adopting the following evolutionary rule (Biondini *et al.* 2004):

$$C_i^{k+1} = \phi_0 C_i^k + \frac{1-\phi_0}{4}\sum_{j=1}^{2}(C_{i-1,j}^k + C_{i+1,j}^k), \qquad (25)$$

where the discrete variable $C_i^k = C(\mathbf{x}_i, t_k)$ represents the concentration of the aggressive agent at time t_k in the cell i of the automaton located at point $\mathbf{x}_i = (y_i', z_i')$ of the cross-section, $C_{i\pm1,j}^k$ is the concentration in the adjacent cells $i \pm 1$ in the direction $j = 1, 2$, and ϕ_0 is a suitable evolutionary coefficient related to the rate of mass diffusion. Moreover, to regulate the process according to a given diffusivity D, a proper discretization in space and time should be chosen in such a way that the grid dimension Δx and the time step Δt satisfy the following relationship:

$$D = \frac{1-\phi_0}{4}\frac{\Delta x^2}{\Delta t}. \qquad (26)$$

The deterministic value $\phi_0 = 1/2$ usually leads to good accuracy of the automaton. However, this evolutionary coefficient must be modelled as a random variable to take into account the stochastic effects in the diffusion process and the corresponding coupling effects between the diffusion process and the mechanical behaviour (Biondini *et al.* 2004).

3.2 *Modelling of structural damage*

Structural damage is modelled by introducing a degradation law of the effective resistant area for both the concrete matrix $A_c = A_c(t)$ and the steel bars $A_s = A_s(t)$:

$$\mathrm{d}A_c(t) = [1-\delta_c(t)]\mathrm{d}A_{c0}, \quad \mathrm{d}A_s(t) = [1-\delta_s(t)]\mathrm{d}A_{s0}, \qquad (27)$$

where the subscript '0' denotes the undamaged state at the initial time $t = t_0$, and the dimensionless functions $\delta_c = \delta_c(t)$ and $\delta_s = \delta_s(t)$ represent *damage indices* that provide a direct measure of the damage level within the range [0; 1]. The time evolution of these indices clearly depends on the corresponding evolution of the diffusion process.

The damaging processes in concrete structures undergoing diffusion are, in general, very complex. Moreover, the available information about environmental agents and material characteristics is usually not sufficient for detailed modelling. However, despite such complexities, very simple degradation models can often be successfully adopted. In the following, the damage indices $\delta_c = \delta_c(\mathbf{x}, t)$ and $\delta_s = \delta_s(\mathbf{x}, t)$ at a point $\mathbf{x} = (y', z')$ of the cross-section are correlated to the diffusion process by assuming, for both materials, a linear relationship between the rate of damage and the concentration $C = C(\mathbf{x}, t)$ of the aggressive agent (Biondini *et al.* 2004):

$$\frac{\partial\delta_c(\mathbf{x},t)}{\partial t} = \frac{C(\mathbf{x},t)}{C_c\Delta t_c}, \quad \frac{\partial\delta_s(\mathbf{x},t)}{\partial t} = \frac{C(\mathbf{x},t)}{C_s\Delta t_s}, \qquad (28)$$

where C_c and C_s represent the values of constant concentration $C(\mathbf{x}, t)$ that lead to complete damage of the concrete and steel after the time periods Δt_c and Δt_s respectively. In addition, the initial conditions $\delta_c(\mathbf{x}, t_{cr}) = \delta_s(\mathbf{x}, t_{cr}) = 0$ with $t_{cr} = \max\{t \mid C(\mathbf{x}, t) \le C_{cr}\}$ are assumed, where C_{cr} is a critical threshold of concentration.

3.3 *Nonlinear structural analysis*

The previous general criteria are now applied to the time-variant nonlinear analysis of deteriorating reinforced concrete cross-sections. By assuming the linearity of the concrete strain field and neglecting the bond-slip of reinforcement, the vectors of the stress resultants $\mathbf{r}=\mathbf{r}(t)=[n\ m_z\ m_y]^T$ and of the global strains $\mathbf{e}=\mathbf{e}(t)=[\varepsilon_0\ \chi_z\ \chi_y]^T$ are then related, at each time instant t, as follows:

$$\mathbf{r}(t) = \mathbf{S}(t)\mathbf{e}(t). \tag{29}$$

The stiffness matrix $\mathbf{S}(t)=\mathbf{S}_c(t)+\mathbf{S}_s(t)$ is derived by integration over the area of the composite cross-section, or by assembling the following contributions of concrete and steel:

$$\mathbf{S}_c(t) = \int_{A_c} E_c(\mathbf{x},t)\ \mathbf{b}(\mathbf{x})^T\ \mathbf{b}(\mathbf{x})[1-\delta_c(\mathbf{x},t)]\mathrm{d}A, \tag{30}$$

$$\mathbf{S}_s(t) = \sum_m E_{\mathrm{sm}}(t)\ \mathbf{b}_m^T\mathbf{b}_m[1-\delta_{\mathrm{sm}}(t)]A_{\mathrm{sm}}, \tag{31}$$

where the symbol 'm' refers to the mth steel bar located at $\mathbf{x}_m=(y'_m,\ z'_m)$, $E_c=E_c(\mathbf{x},\ t)$ and $E_{\mathrm{sm}}=E_{\mathrm{sm}}(t)$ are the secant moduli of the materials, and $\mathbf{b}(\mathbf{x})=[1\ \ -y'\ \ z']^T$ (Biondini *et al.* 2004). It is worth noting that the vectors **r** and **e** have to be considered as total or incremental quantities depending on the nature of the stiffness matrix **S**, which depends on the type of formulation adopted (i.e. secant or tangent) for the generalized moduli of the materials.

Based on this model, the time-variant resistance domain $f(n,\ m_z,\ m_y)\le 0$ of the cross-section can be evaluated with reference to a concrete compression strength properly modified through a suitable effectiveness factor (Nielsen 1999). In addition, the limited ductility of the materials can also be taken into account. For these reasons, these domains can be used as reliable yielding criterion in the limit analysis problem. In particular, the functions $\mathbf{N}=\mathbf{N}(t)$ and $\mathbf{k}=\mathbf{k}(t)$ can be defined by a stepwise approximation of the resistance curves $f(n,\ m_z,\ m_y)=0$, with $m_z=m$ and $m_y=0$ for the planar case.

4. Probabilistic prediction of structural lifetime

From a deterministic point of view, a structure is denoted as safe if the load multiplier λ is no larger than its collapse value $\lambda_c=\lambda_c(t)$. Because of the uncertainties involved in the problem, the quantity λ_c has to be considered as a random variable or process and a measure of structural safety is realistically possible only in probabilistic terms. In particular, by denoting $\hat{\lambda}_k$ an outcome of the random variable $\lambda_{ck}=\lambda_c(t_k)$, the probability of failure at prescribed time instants $t=t_k$ can be evaluated by the integration of the density function $f_{\lambda_c}(\hat{\lambda}_k)$ within the failure domain $D_k=D(t_k)=\{\hat{\lambda}_k\,|\,\hat{\lambda}_k\le\lambda\}$:

$$P_F(t_k) = P[\lambda(t)\ge\lambda_{ck}(t)] = \int_D f_{\lambda_c}(\hat{\lambda}_k)\mathrm{d}\hat{\lambda}. \tag{32}$$

This formulation leads to a stochastic programming problem (Gavarini 1969), which is usually very expensive to solve. Moreover, in practice, the density distribution of λ_c is not known, and at most some information is available only about a set of N basic random variables $\mathbf{X}=[X_1\ X_2,\ldots,X_N]^T$ that defines the structural problem at the initial time $t=t_0$ (e.g. structural geometry **H**, mechanical and geometrical properties of the cross-sections **N** and **k**, dead $\mathbf{f}_0$ and live **f** loads). In addition, in concrete design the levels of verification are usually formulated in terms of functions of random variables $\mathbf{Y}=\mathbf{Y}(\mathbf{X})$ that describe the structural response at each time instant $t=t_k$ (e.g. stress resultants **r**, global strains **e**, etc.), and such derivation is generally only available in an implicit form. A numerical approach is then required and the reliability analysis can be performed by Monte Carlo simulations (Biondini 2000, Biondini *et al.* 2006).

Based on the probabilistic formulation of the time-variant limit analysis problem, the actual lifetime T of the structure associated with a prescribed target reliability level, for example expressed in terms of acceptable values of the probability of failure P_F*, can finally be evaluated as:

$$T = \min\left\{(t-t_0)|P_F\le P_F^*, \forall t\ge t_0\right\}, \tag{33}$$

where t_0 is the time associated with the end of the construction phase.

5. Application to a concrete arch bridge

5.1 *The arch bridge*

The reinforced concrete arch bridge over the Corace river in Italy is now considered. The structural model, shown in figure 4, refers to the data presented in Galli and Franciosi (1955) and Ronca and Cohn (1979). The arch has a rectangular cross-section with nominal dimensions $d_y=0.57$ m and $d_z=6.00$ m, and it is reinforced with $45+45=90$ steel bars, each having a nominal diameter $\varnothing=28$ mm (figure 5(b)). The beam has a two-cellular cross-section with main nominal dimensions $d_y=2.00$ m and $d_z=6.00$ m (figure 5(a)). The distribution of the reinforcement along the beam refers to the subdivision shown in figure 6, and is given in table 1. The structure is subjected to a set of dead loads g and to a live load p (see figure 4). These distributed loads are replaced by statically equivalent concentrated loads, 12 for each span of the girder and 6 for each span of the arch.

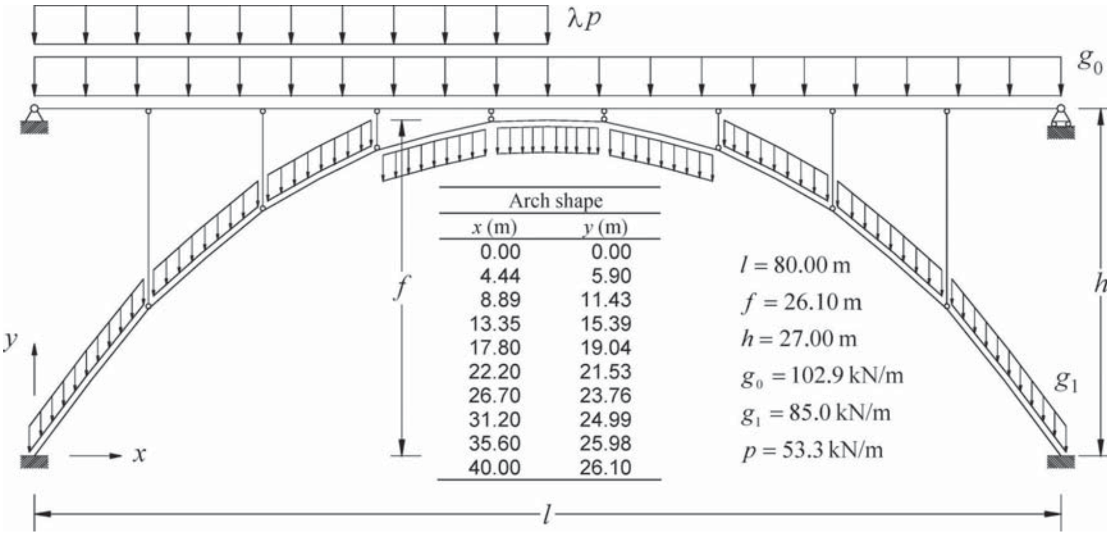

Figure 4. Arch bridge (nominal structure): overall dimensions and loading condition.

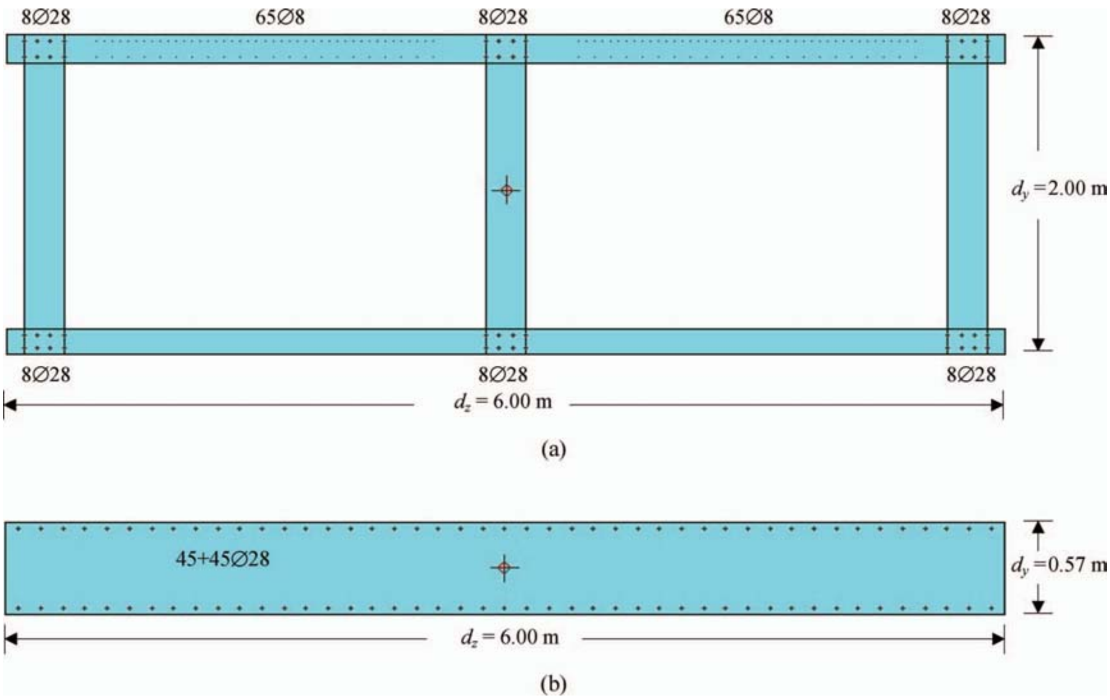

Figure 5. Details of the cross-sections: (a) beam (middle span), and (b) arch.

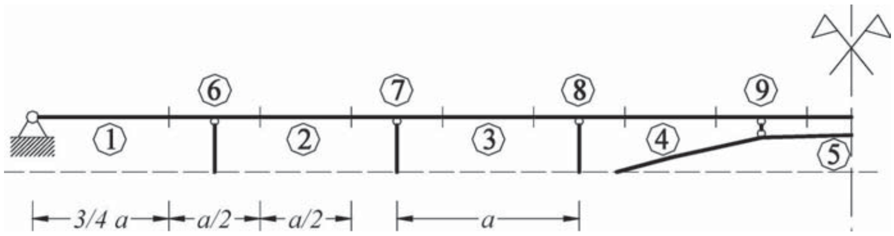

Figure 6. Beam segments having different reinforcement (see table 1).

Table 1. Distribution of the top A_s' and bottom A_s reinforcement along the beam (see figure 6). $n\varnothing d = n$ steel bars with diameter d [mm].

Span	1	2	3	4	5	6	7	8	9
A_s'	21∅28 130∅8	48∅28 130∅8	42∅28 130∅8	30∅28 130∅8	24∅28 130∅8	48∅28 130∅8	48∅28 130∅8	45∅28 130∅8	33∅28 130∅8
A_s	21∅28	30∅28	42∅28	24∅28	24∅28	21∅28	36∅28	27∅28	24∅28

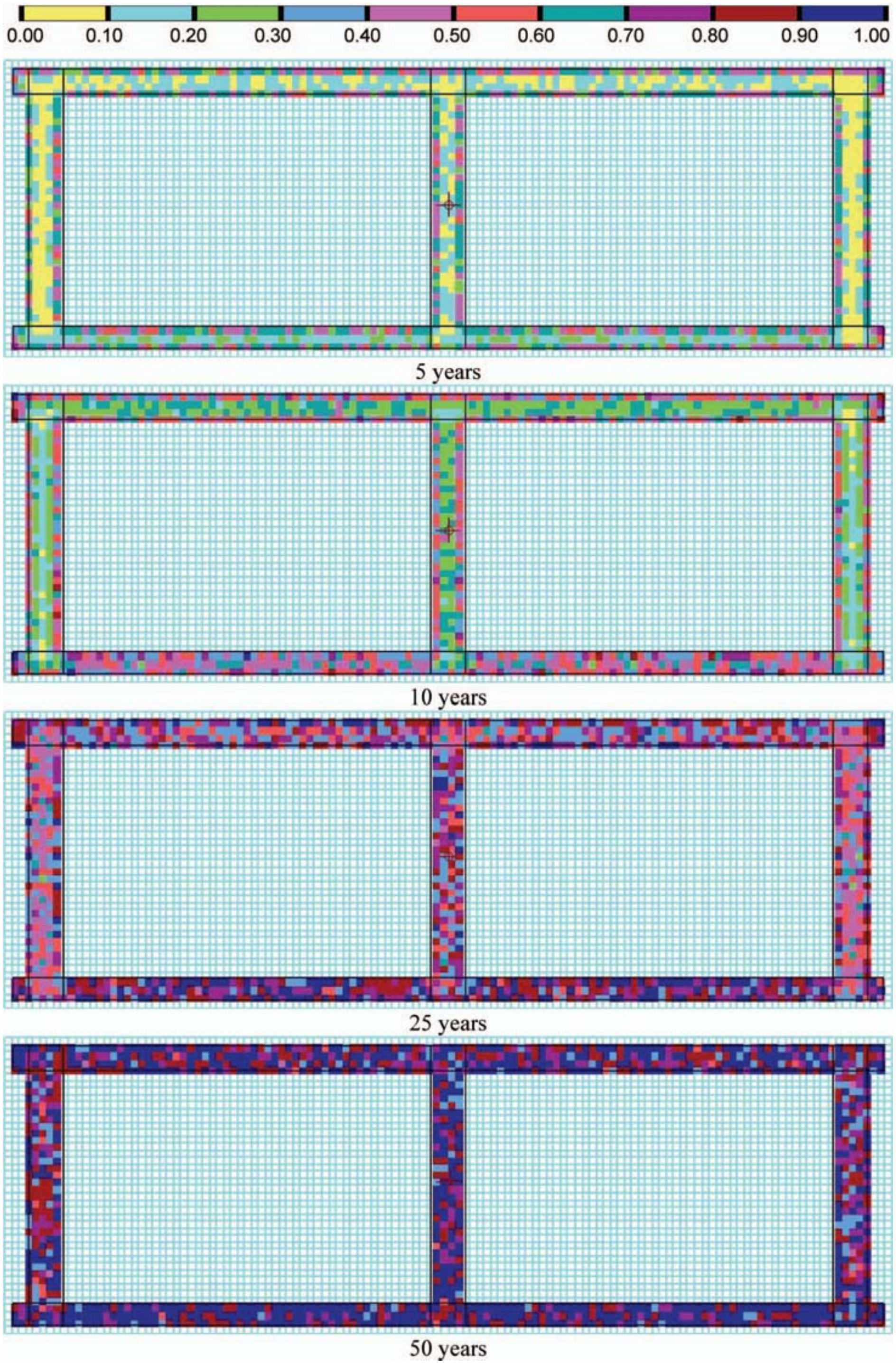

Figure 7. Maps of concentration $C(\mathbf{x}, t)/C_0$ of the aggressive agent in a cross-section of the beam after 5, 10, 25, and 50 years from the initial time of diffusion penetration.

For concrete, the stress–strain diagram is described by the Saenz law in compression and by an elastic perfectly plastic model in tension, with the following nominal parameters: effective compression strength $f_c = -30$ MPa; tension strength $f_{ct} = 0.25|f_c|^{2/3}$; initial modulus $E_{c0} = 9500|f_c|^{1/3}$; peak strain in compression $\varepsilon_{c0} = -0.20\%$; strain limit in compression $\varepsilon_{cu} = -0.35\%$; and strain limit in tension $\varepsilon_{ctu} = 2f_{ct}/E_{c0}$. For steel, the stress–strain diagram is described by an elastic perfectly plastic model in both tension and compression, with the following nominal parameters: yielding strength $f_{sy} = 300$ MPa; elastic modulus $E_s = 206$ GPa; and strain limit $\varepsilon_{su} = 1.00\%$. With reference to a nominal diffusivity coefficient $D = 10^{-11}$ m^2 s^{-1}, the cellular automaton is defined by a grid dimension $\Delta x = 50.2$ mm and a time step $\Delta t = 1$ year. The aggressive agent is assumed to be located with concentration $C(t) = C_0$ along the free edges of the cross-sections. Damage rates are assumed to be defined by the nominal values $C_{cr} = 0$, $C_c = C_s = C_0$, $\Delta t_c = 25$ years and $\Delta t_s = 50$ years.

5.2 *Deterministic time-variant limit analysis*

The diffusion process for the *nominal scenario* is highlighted in figures 7 and 8, which show the maps of concentration $C(\mathbf{x}, t)/C_0$ of the aggressive agent at different time instants in the beam and the arch respectively. The mechanical damage induced by diffusion can be evaluated from the diagrams in figures 9 and 10, which show the time evolution of the resistance bending moments of the axially unloaded beam (figure 9) and of the resistance curves $f(n, m) = 0$ of the arch (figure 10(a)), idealized at each time instant by a four-sides stepwise linearization (figure 10(b)). The deterioration of the five supporting walls, simply compressed, is not investigated since they are assumed as not critical with respect to collapse.

Figure 11 shows the results of the limit analysis carried out for the nominal scenario at the initial time and after 50 years of lifetime. These results highlight that the time evolution of damage leads to a significant variation of the collapse multiplier, which decreases from $\lambda_c = 4.28$ to $\lambda_c = 1.42$, as well as to a noteworthy redistribution of the internal stress resultants and a consequent modification of the collapse mechanism. Figure 12 shows the time evolution of the critical cross-sections where the generalized plastic hinges develop at the collapse. Despite the fact that the location of the plastic hinges tend to be nearly constant over time, local modifications of the plastic strain distribution arise after about 10 years in the arch, and after about 45 years in both the beam and the arch.

5.3 *Probabilistic analysis and lifetime prediction*

The probabilistic model assumes as random variables the location (x_h, y_h) of the nodal connections between the structural elements, the material strengths f_c and f_{sy}, the

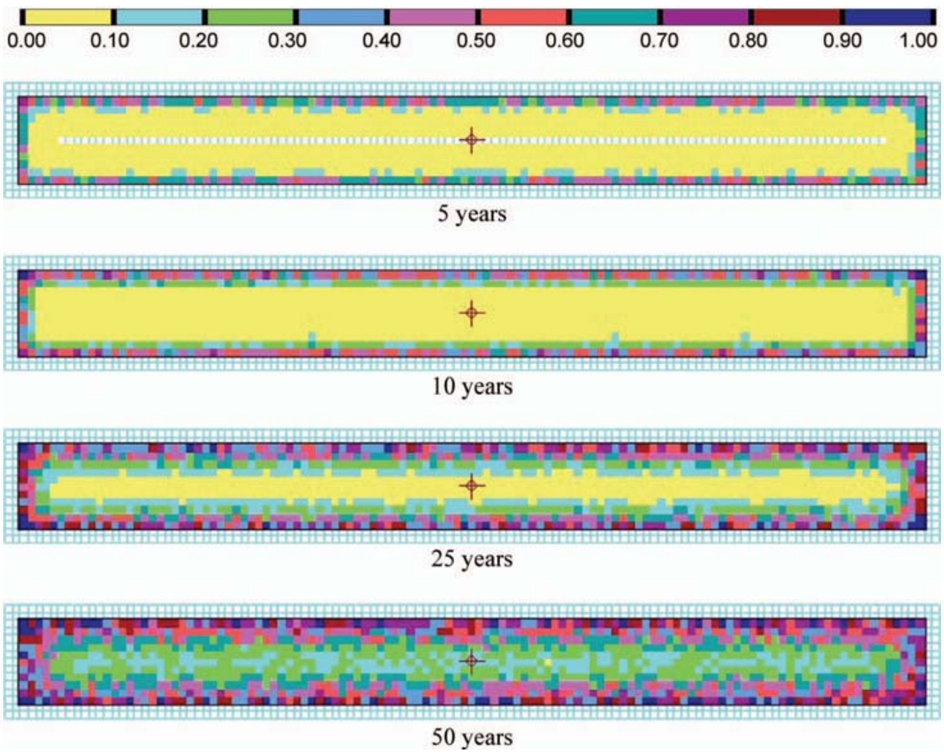

Figure 8. Maps of concentration $C(\mathbf{x}, t)/C_0$ of the aggressive agent in a cross-section of the arch after 5, 10, 25, and 50 years from the initial time of diffusion penetration.

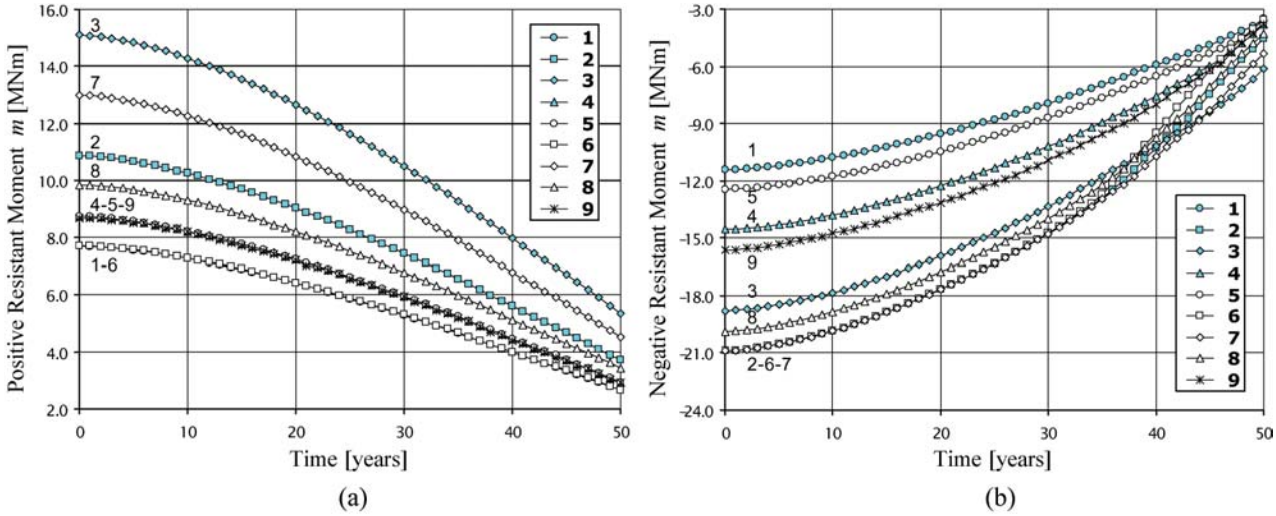

Figure 9. Time evolution of the (a) positive, and (b) negative resistance bending moments of the beam (see table 1).

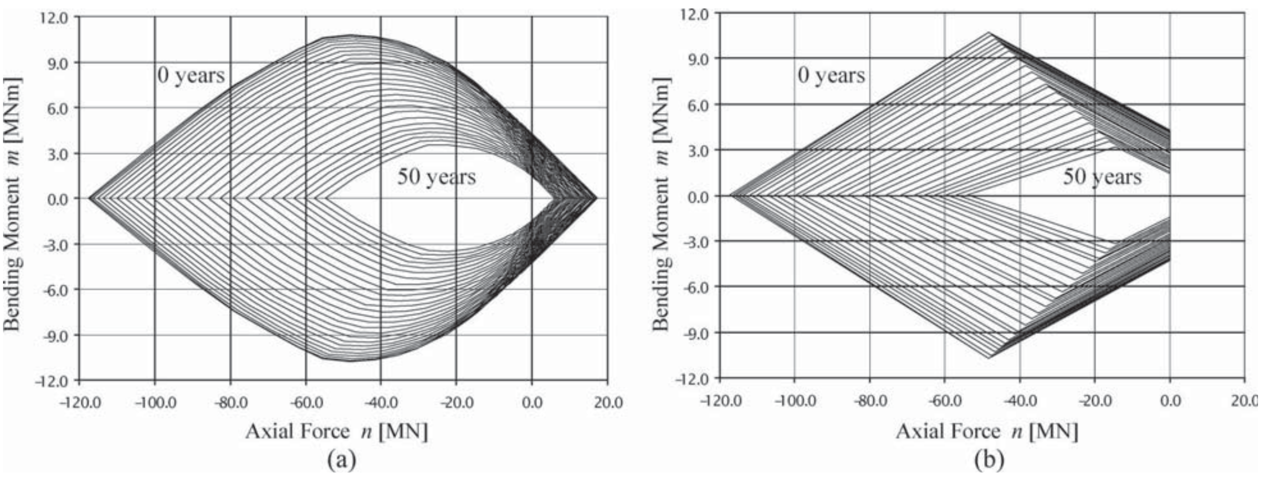

Figure 10. Time evolution of (a) the axial force – bending moment interaction diagram for the arch, and (b) its linearization.

coordinates (y'_p, z'_p) of the nodal points $p = 1, 2, \ldots$ that define the two-dimensional model of the concrete cross-sections, the coordinates (y'_m, z'_m) and the diameter $\varnothing_m$ of the steel bars $m = 1, 2, \ldots$, the diffusivity coefficient D, the damage rates $q_c = (C_c \Delta t_c)^{-1}$ and $q_s = (C_s \Delta t_s)^{-1}$, and the loads g and p in each beam element. A set of random variables is associated with each element of the structural model. These variables are considered to be statistically independent and are assumed to have the probabilistic distribution with the mean μ and standard deviation σ values listed in table 2.

Based on this model, a probabilistic time-variant limit analysis is carried out by using a set of about 28000 Monte Carlo simulations (1000 for each cross-section of the beam, 1000 for the cross-section of each span of the arch, and 1000 for the global analysis of the structure). The time evolution of the collapse multiplier during the first 50 years of lifetime is shown in figure 13(a) for a selected sample of these simulations. In particular, figure 13(a) compares the nominal scenario with the scenarios associated with the minimum and maximum values of the collapse multiplier obtained at the initial time (min(0) and max(0)) and at the end of the structural lifetime (min(50) and max(50)), as well as with the scenarios characterized by the maximum and minimum variation of the collapse multiplier over the investigated lifetime period (max(0 – 50) and min(0 – 50)). With reference to the whole sample, figure 13(b) shows the time

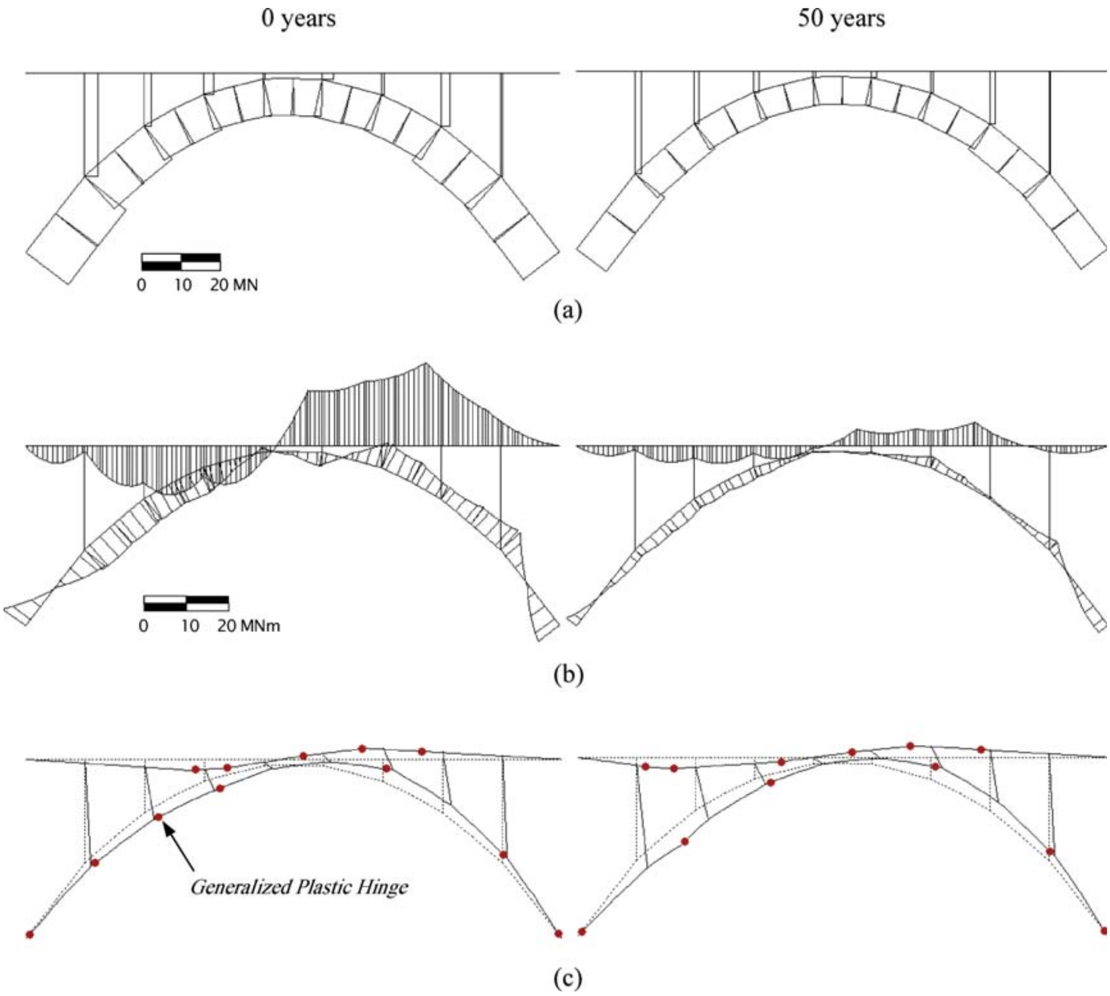

Figure 11. Limit analysis for the nominal scenario at the initial time of construction ($\lambda_c = 4.28$) and after 50 years of lifetime ($\lambda_c = 1.42$): (a) axial force, and (b) bending moment diagrams at the collapse; (c) collapse mechanism.

evolution of the statistical parameters (mean value μ, standard deviation σ, minimum and maximum values) of the collapse multiplier. The validity of the simulation results with respect to the sample size N is highlighted by figure 14, which allows the expected convergence towards stable values of both the mean μ and deviation σ/μ to be verified. A sensitivity analysis of the relative importance of each random variable in the probabilistic model has been presented in Biondini and Frangopol (2006).

The results of this simulation can be used to compute, at each point in time, the probability of failure for given deterministic target levels of the structural performance indicators, as shown by the probability curves in figure 15(a). These curves allow the time-variant reliability of the cross-section with respect to the required performance to be assessed. Moreover, based on these probability curves, the lifetime T associated with given target reliability levels P^* can be computed as a function of the expected values of the live load multiplier, as shown in figure 15(b). These curves allow the remaining lifetime, which can be assured under prescribed reliability levels without maintenances, to be assessed.

6. Conclusions

The structural lifetime of deteriorating concrete frames with respect to the ultimate limit state of structural collapse has been investigated. The structural system is considered to be exposed to an aggressive environment and the effects of the damaging process induced by the diffusive attack of external agents are described by the corresponding time evolution of the axial force – bending moment resistance domains. Based on such domains, the time-variant collapse load is computed by means of limit analysis through a

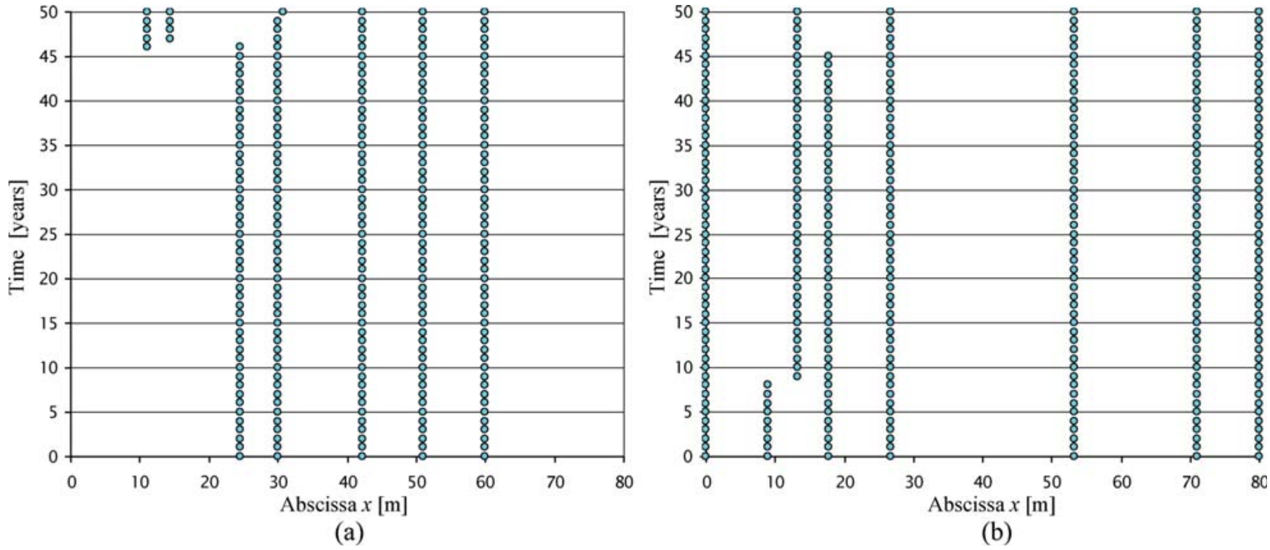

Figure 12. Time evolution of the location of the generalized plastic hinges for the nominal scenario: (a) in the beam, and (b) in the arch.

Table 2. Probability distributions and their parameters (truncated distributions provide non-negative outcomes).

Random Variable ($t = t_0$)	Distribution Type	μ	σ
Coordinates of the nodal points, (x_h, y_h)	Normal	$(x_h, y_h)_{\text{nom}}$	50 mm
Concrete strength, f_c	Lognormal	$f_{c,\text{nom}}$	5 MPa
Steel strength, f_{sy}	Lognormal	$f_{sy,\text{nom}}$	30 MPa
Coordinates of the nodal points, (y'_p, z'_p)	Normal	$(y'_p, z'_p)_{\text{nom}}$	5 mm
Coordinates of the steel bars, (y'_m, z'_m)	Normal	$(y'_m, z'_m)_{\text{nom}}$	5 mm
Diameter of the steel bars, $\varnothing_m$	Normal-truncated	$\varnothing_{m,\text{nom}}$	$0.10\varnothing_{m,\text{nom}}$
Diffusivity coefficient, D	Normal-truncated	D_{nom}	$0.10\ D_{\text{nom}}$
Concrete damage rate, $q_c = (C_c \Delta t_c)^{-1}$	Normal-truncated	$q_{c,\text{nom}}$	$0.30\ q_{c,\text{nom}}$
Steel damage rate, $q_s = (C_s \Delta t_s)^{-1}$	Normal-truncated	$q_{s,\text{nom}}$	$0.30\ q_{s,\text{nom}}$
Dead loads, g	Normal-truncated	g_{nom}	$0.10\ g_{\text{nom}}$
Live load, p	Normal-truncated	p_{nom}	$0.40\ p_{\text{nom}}$

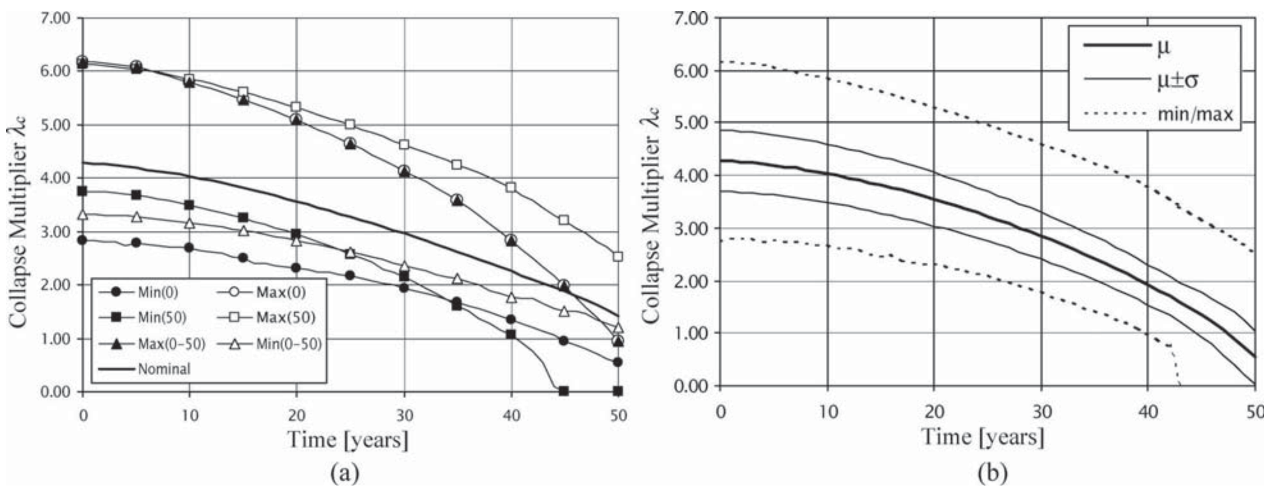

Figure 13. Time evolution of the collapse multiplier λ_c: (a) comparison among selected simulations, and (b) mean μ (thick line), standard deviation σ from the mean μ (thin lines), minimum and maximum values (dotted lines).

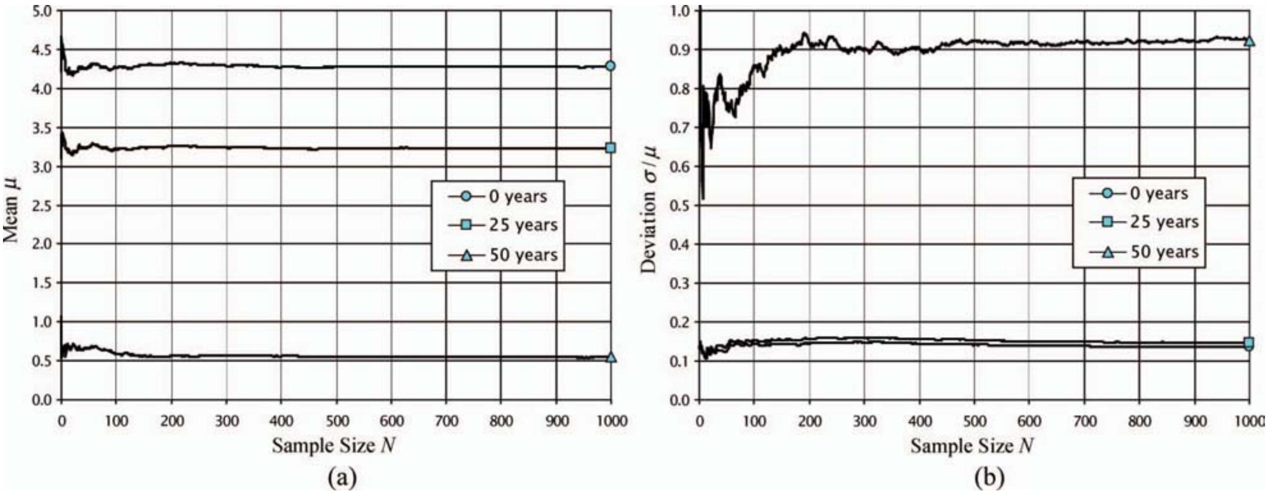

Figure 14. Evolution during the simulation of (a) mean value μ, and (b) deviation σ/μ of the collapse multiplier after 0, 25, and 50 years of lifetime.

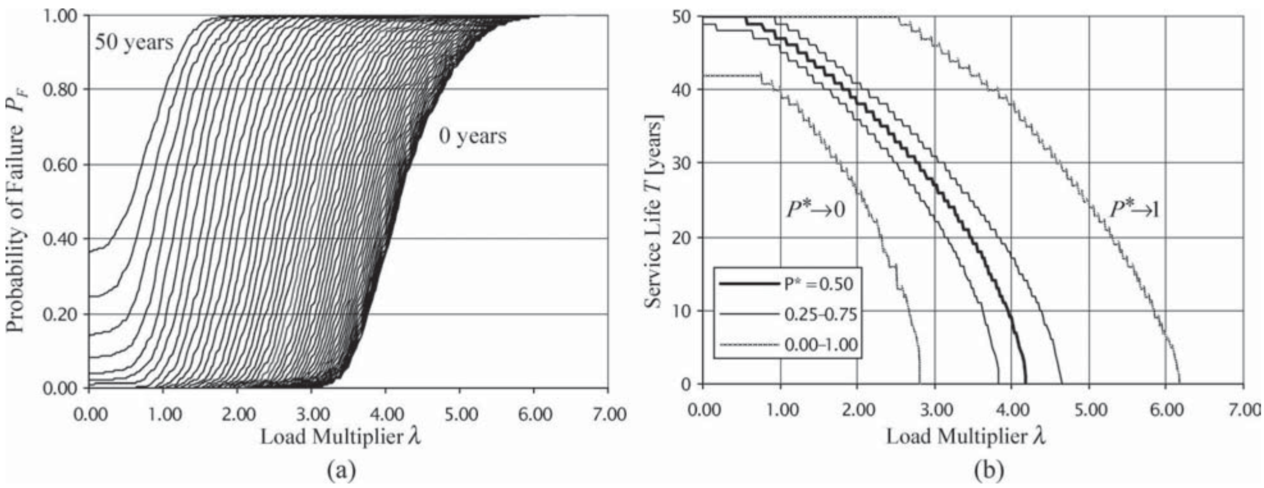

Figure 15. Time evolution of the collapse multiplier λ_c: (a) probability curves during the first 50 years of service life ($\Delta t = 2$ years), and (b) service life T associated with given values of the probability of failure P^* versus given target levels of the load multiplier λ.

systematic approach that considers axial force and bending moments as active and interacting generalized plastic stresses. The randomness of the main structural parameters is taken into account by Monte Carlo simulations, and both the time-variant probability of failure, as well as the expected structural lifetime associated to a prescribed reliability level are evaluated.

The application to the collapse reliability analysis and lifetime prediction of a reinforced concrete arch bridge shows that the proposed approach predicts the time-variant structural resistance with respect to a given demand or, conversely, the corresponding remaining lifetime that can be assured under prescribed reliability levels without maintenance.

The accuracy of the results mainly depends on the values of the material parameters that define both the diffusive and damage processes. Further developments aimed to achieve a proper calibration of the material parameters and of their probabilistic distributions are required. However, despite the necessity of such developments, the proposed approach is proven to represent a powerful engineering tool for the reliability assessment and lifetime prediction of deteriorating reinforced concrete framed structures with respect to the collapse.

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